

COALGEBRAIC NOTIONS OF SIMULATION, BISIMULATION AND RELATORS

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The content of the talk:

- Intuitive introduction to coalgebra and (bi)simulation
- Reviewing basic definitions of coalgebra and (bi)simulation
- Our motivation: To ease proving program equivalence
- Relator-based notions
- Span-based notions

WHY DOES EQUIVALENCE OF SYSTEMS MATTER?

A basic but fundamental question

When can we say that two systems *behave the same*?

This question is everywhere, often silently assumed:

🔧 **Hardware redesign.** A chip redesigned for cheaper production must still compute exactly what the old one did.

</> **Software upgrades.** A bank replaces its backend — from the customer's view (balances, transactions), nothing should change.

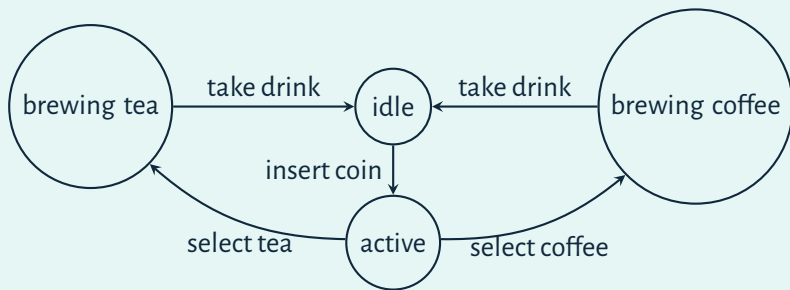
The catch

“Behaving the same” is intuitive, but making it *precise* and *checkable* is surprisingly subtle.

FROM “SYSTEM” TO MATHEMATICS

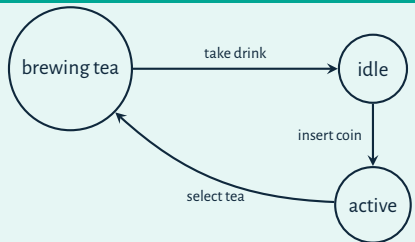
We model systems as mathematical notions like **labeled transition systems**: a set of states, together with a labeled transition relation between them.

Example: a vending machine

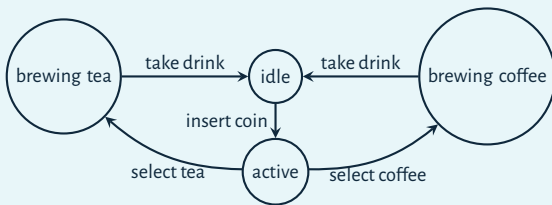
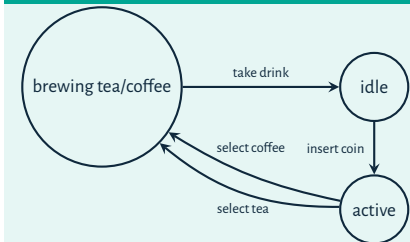


SIMULATION AND BISIMULATION

Similar to



Bisimilar with



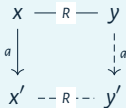
MATHEMATICAL DEFINITIONS

Labelled Transition Systems

A *labelled transition system* is a triple (X, A, T) , where $T \subseteq X \times A \times X$. We write $x \xrightarrow{a} x'$ for $(x, a, x') \in T$.

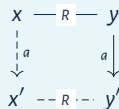
Simulation

A relation $R \subseteq X \times Y$ is a *simulation* from (X, A, T) to (Y, A, S) whenever:



Bisimulation

A simulation relation like R is a *bisimulation* whenever:



Similarity and Bisimilarity

The *similarity relation* $\preceq \subseteq X \times Y$ is the greatest simulation relation. Equivalently

$$x \preceq y \iff \exists R. xRy,$$

where R is a simulation.

Bisimilarity is defined similarly, where R is a bisimulation, and it is shown with $\sim$.

COALGEBRAS: A UNIFORM VIEW OF TRANSITION SYSTEMS

Coalgebras

Fix an endofunctor

$$F : \mathbf{C} \rightarrow \mathbf{C}.$$

An F -coalgebra is a pair (X, γ) , consisting of an object X , and a map

$$\gamma : X \rightarrow FX.$$

Different choices of F describe different transition systems.

Our Goal

Develop **simulation**, **bisimulation** once, for all coalgebras.

Different kinds of systems

- Labeled Transition Systems
($FX = \mathcal{P}(A \times X)$)
- Kripke frames ($FX = \mathcal{P}X$)
- Operational Semantics of Programming Languages (*Abstract HO-GSOS, and Big-step SOS!*)
- Markov Chains ($FX = (\mathcal{D}X)^A$)
- Mealy Machines ($FX = (O \times X)^I$)
- $\vdots$

TOWARDS PROGRAM EQUIVALENCE

We have developed a coalgebra (T, γ) that gives the big-step semantics for an arbitrary programming language in a specified family of them, called *abstract HO-GSOS*.

Main Motivation

We want to prove that for terms t and s , and a context C :

$$t \sim s \Rightarrow C[t] \sim C[s]$$

- In the literature, this is often proved using *Howe's method*.
- The method, applies a closure on the bisimilarity relation on T .
- Then one should prove that the result is a simulation.
- The closure preserves symmetry, and symmetric simulation is a bisimulation in traditional definitions.
- We have found out that it is not always the case.
- That is why we want to know when exactly a symmetric simulation is a bisimulation.

RELATORS AND SIMULATIONS

To define coalgebraic simulation, relators are invented.

Relator

Assuming $F : \mathbf{Set} \rightarrow \mathbf{Set}$ is a functor, an F -relator is a monotone map that sends a morphism of $\mathbf{Rel}$ that is a relation $X \rightharpoonup Y$ to a relation $FX \rightharpoonup FY$.

R-Simulation

For an F -relator R , a relation $r : X \rightharpoonup Y$ is a R -simulation from a coalgebra (X, α) to a coalgebra (Y, β) if $r \subseteq \beta \cdot Rr \cdot \alpha^{-1}$.

R-Similarity

R -similarity is the greatest R -simulation.

Example: Trivial Relator

For an arbitrary set endofunctor F , the relator $\mathbf{K}$ takes every relation $r : X \rightharpoonup Y$ to $FX \times FY$.

But it is a bad relator because every relation is a $\mathbf{K}$ -bisimulation!

Example: Egli-Milner Relator

The relator $\mathbf{L}$ is a $\mathcal{P}$ -relator defined as:

$$\mathbf{L}r = \{(S, T) \mid x \in S \Rightarrow \exists y \in T, x \mathrel{r} y\}$$

It gives us the traditional definition for simulation on Kripke frames.

SYMMETRIC RELATORS

R-bisimulation (R-bisimilarity)

An R-simulation (similarity) is a bisimulation (bisimilarity), if R is *symmetric*.

Symmetrization of a Relator

Symmetrization of a relator R is defined as:

$$\hat{R} = Rr \cap (Rr^{-1})^{-1}$$

Symmetric Relators

A relator R is symmetric if $R = \hat{R}$

Egli-Milner is not symmetric.

BARR RELATORS

Example: Barr Relators

Given a set endofunctor F , and r is a relation that $r = \pi_2 \cdot \pi_1^{-1}$ (π_1 and π_2 are functions) the Barr relator $\bar{F}$ is defined as $\bar{F}r = \bar{F}\pi_2 \cdot (\bar{F}\pi_1)^{-1}$.

The symmetrization of the Egli-Milner relator is a Barr relator, i.e.,

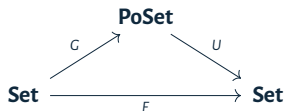
$$\hat{\mathbf{L}} = \bar{\mathcal{P}}$$

Barr relators are nicely behaved relators.

So, we always have bisimulations with Barr relators.

LAX BARR RELATORS

An ordering of a set endofunctor F is a functor G that commutes in the following diagram:



Meaning that for every X , there is a poset $(FX, \sqsubseteq)$.

For example $(\mathcal{P}X, \sqsubseteq)$ for every set X .

With orderings we define *lax Barr relators*.

Lax Barr relators are different abstractions of the Egli-Milner relator.

- Left-lax: $\sqsubseteq \cdot \bar{F}r$
- Right-lax: $\bar{F} \cdot \sqsubseteq$
- Bi-lax: $\sqsubseteq \cdot \bar{F}r \cdot \sqsubseteq$
- Mid-lax: $F\pi_1 \cdot \sqsubseteq \cdot (F\pi_2)^{-1}$

Liftability

An ordering on a set endofunctor F is called *liftable*, if

$$h \sqsubseteq Fg \cdot k \Rightarrow \exists k', h = Fg \cdot k' \ \& \ k' \sqsubseteq k.$$

An ordering and liftability can be defined for an arbitrary category as well.

SYMMETRIZATION OF LAX BARR RELATORS

Having liftability allows us for an arbitrary function g to have:

$$\begin{array}{ccc} FX & \xrightarrow[\sqsubseteq_x]{} & FX \\ Fg \downarrow & & \downarrow Fg \\ FY & \xrightarrow[\sqsubseteq_y]{} & FY \end{array}$$

Proposition

Given a set endofunctor F , with a liftable ordering

1. mid-lax and left-lax relators of F are equal,
2. Assuming the axiom of choice, right-lax and left-lax relators of F are equal,
3. the symmetrizaion of the left-lax relator is *sound* and *complete* with respect to *behavioural equivalence*.

- Although, not every sound and complete relator is a Barr relator.
- We do not know when the symmetrizaion of a Barr relator is a Barr relator.

SPAN-BASED DEFINITIONS

Aczel-Mendler Bisimulation

For coalgebras (X, α) , and (Y, β) a span (R, X, Y) is an *AM-bisimulation*, if there is a *witness* γ that commutes in the following:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & Y \\ \alpha \downarrow & & \downarrow \gamma & & \downarrow \beta \\ FX & \xleftarrow{Fp_1} & FR & \xrightarrow{Fp_2} & FY \end{array}$$

In a *regular category*, if we change FR with $\bar{F}R$ that shows the image of $\langle Fp_1, Fp_2 \rangle$, and the legs of $\bar{F}$ (Barr relator in **Set**), we have an *Hermida-Jacobs* bisimulation.

Every AM-bisimulation is an HJ-bisimulation.

AM-Simulation

We have *AM-simulation*, if we lax the diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & Y \\ \alpha \downarrow & \lrcorner & \downarrow \sigma & \lrcorner & \downarrow \beta \\ FX & \xleftarrow{Fp_1} & FR & \xrightarrow{Fp_2} & FY \end{array}$$

- Every AM-(bi)simulation is an HJ-(bi)simulation.
- Assuming AoC every HJ-bisimulation is an AM-bisimulation.
- Every HJ-bisimulation for a functor of form $\mathcal{P}F$ in a topos is an AM-bisimulation.

SYMMETRIC SIMULATION FOR SPAN-BASED DEFINITIONS

So far we have:

- An example of an $\text{AM}(\text{HJ})$ -simulation that is not an $\text{AM}(\text{HJ})$ -bisimulation.
- An example of an $\text{AM}(\text{HJ})$ -simulation that the witness for $\text{AM}(\text{HJ})$ -simulation does not serve as a witness for $\text{AM}(\text{HJ})$ -bisimulation for a $\mathcal{P}$ -coalgebra in **Set**.

Proposition

Given a symmetric span (R, X) , with the symmetry witness $s : R \rightarrow R$, assuming R is an HJ -simulation on a coalgebra (X, α) with witness σ , then R is an AM -bisimulation iff s is a coalgebra endomorphism on (R, σ) .

- We have a proof that symmetric AM -simulations for coalgebras of $\mathcal{P}$ and the maybe functor on **Set** are AM -bisimulation.
- We want to know when exactly this proposition holds (perhaps liftability matters).

Thank you for your attention! :-)